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Report on 3 and 4-Point Correlation Statistics in the COBE DMR Anisotropy Maps

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1. Summary

As part of the work performed under NASA contract # NAS5-32648, we have computed the 3-point and 4-point correlation functions of the COBE-DMR 2-year and 4-year anisotropy maps. The motivation for this study was to search for evidence of non-Gaussian statistical fluctuations in the temperature maps: skewness or asymmetry in the case of the 3-point function, kurtosis in the case of the 4-point function. Such behavior would have very significant implications for our understanding of the processes of galaxy formation, because our current models of galaxy formation predict that non-Gaussian features should not be present in the DMR maps. The results of our work showed that the 3-point correlation function is consistent with zero and that the 4-point function is not a very sensitive probe of non-Gaussian behavior in the COBE-DMR data.

Our computation and analysis of 3-point correlations in the 2-year DMR maps was published in the Astrophysical Journal Letters, volume 446, page L67, 1995. Our computation and analysis of 3-point correlations in the 4-year DMR maps will be published, together with some additional tests, in the June 10, 1996 issue of the Astrophysical Journal Letters. Copies of both of these papers are attached as an appendix to this report.

2. 3-point Correlations in the COBE-DMR Anisotropy Maps

We have computed the 3-point correlation function of the COBE-DMR 2-year and 4-year anisotropy maps to search for evidence of skewness or asymmetry in the temperature fluctuations. Such behavior would have very significant implications for our understanding of the processes of galaxy formation, because our current models of galaxy formation predict that significant skewness should not be present in the DMR maps.

Our results showed that the 3-point correlation function is consistent with zero and that the fluctuations are consistent with being Gaussian distributed. Moreover, with the improved sensitivity in the 2- and 4-year maps we were able to place significantly tighter upper limits on non-Gaussian fluctuations than previously existed, by nearly an order of magnitude. Our computation and analysis of 3-point correlations in the 2-year DMR maps was published in the Astrophysical Journal Letters, volume 446, page L67, 1995. Our computation and analysis of 3-point correlations in the 4-year DMR maps will be published, together with some additional tests, in the June 10, 1996 issue of the Astrophysical Journal Letters. Copies of both of these papers are attached as an appendix to this report.

3. 4-point Correlations in the COBE-DMR Anisotropy Maps

We have computed a special case of the 4-point correlation function of the COBE-DMR 2-year maps to search for evidence of non-zero kurtosis in the temperature fluctuations: i.e. evidence that the "tails" of the temperature distribution differ significantly from a Gaussian distribution. Our results were inconclusive because of limitations with the DMR data: the level of "cosmic variance" is too high to make this higher-order statistic very useful. In other words, because of COBE's coarse angular resolution, there are not enough independent points in the sky to usefully probe the tails of the temperature distribution. Since the results of this effort were not deemed interesting enough to submit for publication, we summarize the results of our study in some detail here.

The simplest configuration of the 4-point function is one in which the 4 legs of the function are collapsed to two. This configuration depends only on the angular separation between the two legs, and is defined as

$$C_4^{(c)}(\alpha) = \frac{1}{N_\alpha} \sum_{i,j} t_i^2 t_j^2 \tag{1}$$

with

$$N_\alpha = \sum_{i,j} 1$$

where t_i is the temperature in pixel i, and the sum on j is restricted to pixels within angular separation bin α of pixel i. The ensemble average over realizations of the sky temperature is given

by

$$\begin{aligned}\langle C_4(\alpha)\rangle &= \langle \frac{1}{N_\alpha}\sum_{i,j} t_i^2 t_j^2\rangle \\ &= \frac{1}{N_\alpha}\sum_{i,j}\langle t_i^2 t_j^2\rangle\end{aligned}$$

If the temperatures have a Gaussian distribution we may write

$$\begin{aligned}\langle t_i^2 t_j^2\rangle &= \langle t_i^2\rangle\langle t_j^2\rangle + 2\langle t_i t_j\rangle^2 \\ &= \langle C_2(0)\rangle^2 + 2\langle C_2(\alpha)\rangle^2\end{aligned}$$

where $\langle C_2(\alpha)\rangle$ is the ensemble-averaged 2-point correlation function. It follows that

$$\langle C_4(\alpha)\rangle = \langle C_2(0)\rangle^2 + 2\langle C_2(\alpha)\rangle^2$$

The above result suggests that we define a generalized kurtosis statistic as

$$K(\alpha) \equiv C_4(\alpha) - C_2(0)^2 - 2C_2(\alpha)^2 \tag{2}$$

One might expect this statistic to have zero mean for Gaussian distributed temperatures t_i, but that is not the case. The ensemble average of K is given by

$$\begin{aligned}\langle K(\alpha)\rangle &= \langle C_4(\alpha)\rangle - \langle C_2(0)^2\rangle - 2\langle C_2(\alpha)^2\rangle \\ &= \langle C_2(0)\rangle^2 + 2\langle C_2(\alpha)\rangle^2 - \langle C_2(0)^2\rangle - 2\langle C_2(\alpha)^2\rangle\end{aligned}$$

which is, in general, not zero because of fluctuations in the 2-point function. If we define

$$C_2(\alpha) \equiv \langle C_2(\alpha)\rangle + \delta(\alpha)$$

where $\delta(\alpha)$ is the deviation of an individual correlation function from the ensemble average, due, in general to cosmic variance and instrument noise, then, using $\langle\delta(\alpha)\rangle \equiv 0$, we can write

$$\begin{aligned}\langle K(\alpha)\rangle &= \langle C_2(0)\rangle^2 + 2\langle C_2(\alpha)\rangle^2 - \langle(\langle C_2(0)\rangle + \delta(0))^2\rangle - 2\langle(\langle C_2(\alpha)\rangle + \delta(\alpha))^2\rangle \\ &= -\langle\delta(0)^2\rangle - 2\langle\delta(\alpha)^2\rangle\end{aligned}$$

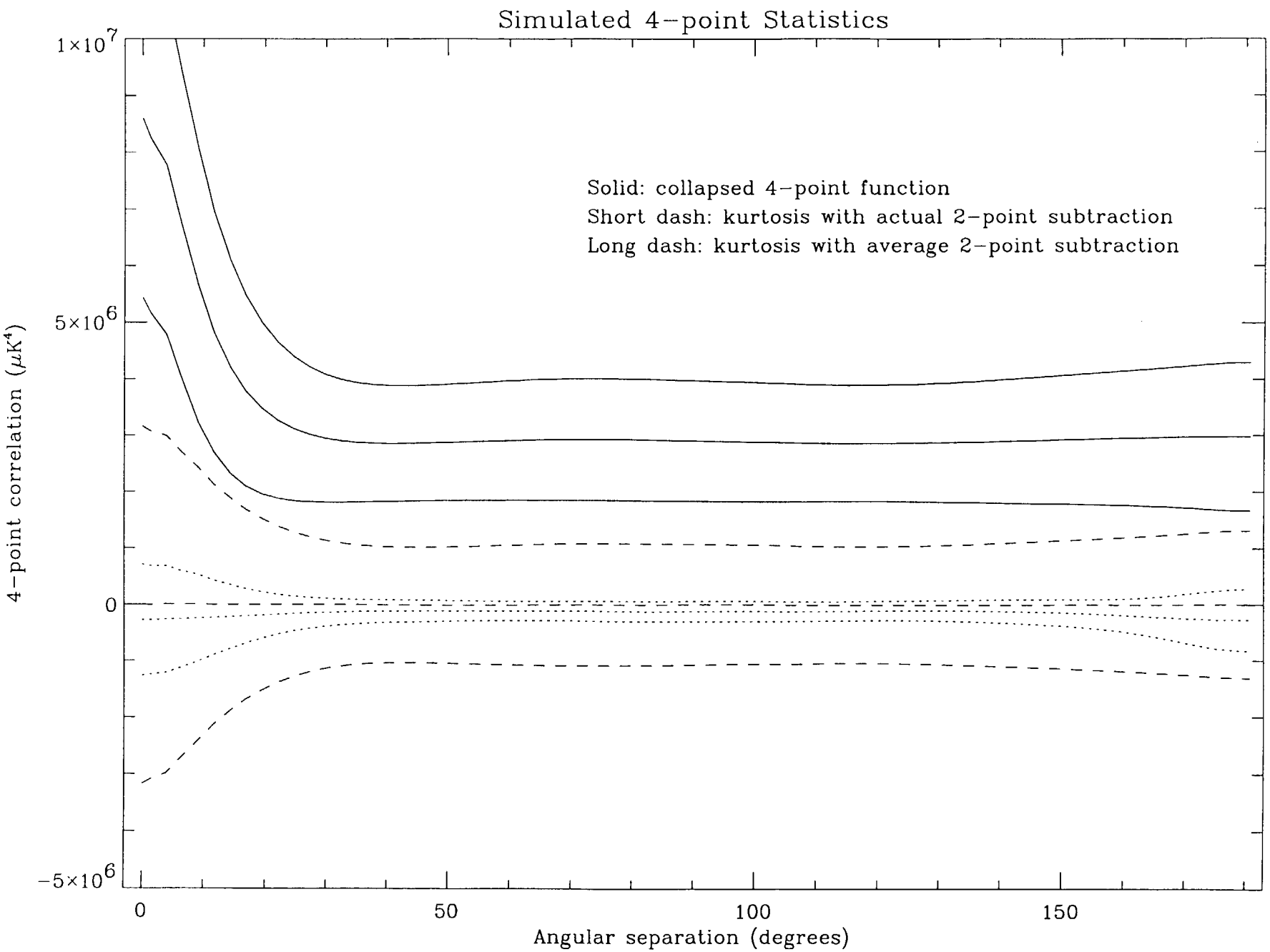
Thus the larger the fluctuations in the 2-point function the more negative the mean of K will be. Because of this non-vanishing mean, it is problematic to interpret the results of a computation of $K(\alpha)$. In principle, one could avoid this by defining a modified kurtosis to be:

$$K'(\alpha) \equiv C_4(\alpha) - \langle C_2(0)\rangle^2 - \langle 2C_2(\alpha)\rangle^2 \tag{3}$$

where the quantity subtracted off is a power of the ensemble-averaged 2-point function. This would have zero mean, but it requires that we know the ensemble-averaged 2-point function, a priori, which we don't.

Figure 1 shows the results of a Monte Carlo simulation of the 4-point function and the generalized kurtosis, as defined above. We simulated 2000 synthetic COBE-DMR sky maps with a Gaussian, scale-invariant, power-law model of CMB anisotropy, and instrument noise. For each realization, we computed the 4-point function, defined in (1), the generalized kurtosis, defined in (2), and the modified generalized kurtosis, defined in (3). Given 2000 realizations of each function we then compute its mean and standard deviation and plot the results as a function of angular separation. The solid lines in the plot show the mean and standard deviation of the 4-point function, $C_4(\alpha)$. Note that the mean follows $\langle C_2(0) + 2C_2(\alpha)\rangle$, as expected, and has a rather large standard deviation, due mostly to cosmic variance. The short dashed lines show the mean and standard deviation of the kurtosis, $K(\alpha)$. Note that the mean is non-zero, as expected, but that the standard deviation is much smaller than with $C_4(\alpha)$. Finally, the long dashed lines show the mean and standard deviation of the modified kurtosis, $K'(\alpha)$. Here the mean is zero (we have the advantage of knowing $\langle C_2(\alpha)\rangle$ because this is a simulation), but the standard deviation is much larger than with $K(\alpha)$.

Even with the most sensitive statistic, $K(\alpha)$, the standard deviation is so large, $\sim 30\,\mu\mathrm{K}^4$, that it is completely insensitive to reasonable deviations from Gaussian statistics. To verify this, we have also run some simulations with "toy" non-Gaussian models in which the distribution of spherical harmonic coefficients, $a_{\ell m}$, were chosen to be very different from Gaussian to see if the 4-point statistic could pick this up. In general, the statistic was very insensitive to changes in the distribution of the $a_{\ell m}$'s, further mitigating the interest of this statistic.


Simulated 4−point Statistics
Solid: collapsed 4−point function
Short dash: kurtosis with actual 2−point subtraction
Long dash: kurtosis with average 2−point subtraction
4−point correlation (μK⁴)
1×10⁷
5×10⁶
0
−5×10⁶
0
50
100
150
Angular separation (degrees)

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